Beam Selection Based on Sequential Competition

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Abstract—We present a novel $M$-ary sequential test for beam selection when knowledge about the SNR operating point is not available. The proposed sequential test adaptively changes the test length (the number of observations) according to the SNR operating point to achieve the desired performance. Moreover, to achieve the same performance in terms of captured signal power, the sequential competition test requires on average less observations (particularly in the lower SNR regime) in comparison to a perfectly tuned fixed length test assuming genie knowledge.

Index Terms—Beam Selection, Sequential Test, Generalized Likelihood Ratio Test

I. INTRODUCTION

Directional data transmission in practical multiple antenna systems can be simply achieved via electronically controllable beamforming networks like Butler Matrix [1]. Such a network provides a set of orthogonal beams (denoted as ‘codebook’) steered into different mainlobe directions.

A crucial question at the receiver side during beam training showed schematically in Fig 1, is how to efficiently select the best beam(s) capturing the highest amount of received power from a given codebook.

State of the art beam-alignment techniques e.g. exhaustive or tree search with hierarchical codebooks, use a fixed length training sequence for evaluation of the best beam i.e. beam selection [2][3][4][5]. A fixed length test can only be optimally designed for one particular Signal to Noise Ratio (SNR). Below this point the decisions become unreliable and above it we take too many observations. However, in most practical applications, the knowledge about SNR is not or only roughly available and the problem needs to be solved over some non-negligible SNR range due to varying operation conditions. This fact motivated us to develop an adaptive variable length approach.

In this letter we present the underlying idea of beam selection based on sequential competition which was introduced in [6]. This algorithm learns on the fly the current statistics of the signals in terms of their amplitudes and noise variances based on the available observations from each beam. In this way the decision on the best beam(s) can be made both as early as possible and adaptively according to the SNR operating point. This becomes particularly interesting when the number of beams from which we can choose, is large and/or the SNR value is low. This scenario is expected to occur frequently during initial acquisition or handover in communication systems using large antenna arrays.

II. ANALOG BEAM SELECTION PROBLEM

Consider the problem of finding the detector for selecting the best beam which captures the highest signal power, from a set of candidate beams each with different main lobe direction. Although the channel model for mmWave systems contain features like multipath with multiple delays and angle of arrivals, we believe that the essential aspects of the detection problem under consideration that arises after correlation to a training sequence, are captured by a flat channel model. This leads to the following complex valued received samples which are observed separately$^2$ under each candidate beam as

$$r_m[n] = A_m s[n] + w_m[n],$$

where $m \in \{1, \ldots, M\}$ and $n \in \{1, \ldots, N\}$ indicate the beam and sample indices. $A_m$ is the combined effective channel and beamforming gain corresponding to beam $m$, that is treated as a deterministic unknown complex amplitude. $w_m[n]$ is complex zero mean white Gaussian noise (WGN) with unknown variance $\sigma^2$ under beam $m$. A pseudo-random sequence$^3$ with $s[n] \in \{\pm 1\}$, variance one and $P\{s[n] = +1\} = P\{s[n] = -1\} = 1/2$ is assumed for training so that $E[s[n]s[n-k]] = \delta[k]$ holds for its autocorrelation sequence.

Considering the vectors of correlated observations $y_m = [s[1]r_{m}[1], \ldots, s[N]r_{m}[N]]$, a fixed length test with length $N$ decides for beam $k$ over all candidate beams, based on minimum variance unbiased (MVU) signal power estimates, if

$$k = \arg\max_{m \in \{1, \ldots, M\}} \{ |y_m y_m^H|/N \},$$

where $(.)^H$ and $|.|$ denote conjugate transpose and absolute value operations, respectively.

$^2$Depending on the number of RF chains with respect to number of antennas, the observations from different beams at each $n$ can be made in parallel, serial or in a combined fashion.

$^3$Assuming a set of sequences with good auto- and cross-correlation properties so that different users can be assigned to different sequences allows a generalization to multipath and multiuser scenarios.
Defining the maximum gain as $|A_{\text{max}}| = \max(|A_1|, \ldots, |A_M|)$, the above detector results in the following normalized average loss of the signal magnitude

$$\bar{\ell} = 1 - \frac{1}{|A_{\text{max}}|} \sum_{m=1}^{M} P\{k = m|y_1, \ldots, y_M|A_m\} ,$$

where $P\{k = m|y_1, \ldots, y_M\}$ denotes the probability of selecting beam $m$ after observing the sequences $y_1, \ldots, y_M$.

The problem that arises in the design of the training sequence (or the detector) is to choose a specific length $N$ to achieve a certain performance in terms of $\bar{\ell}$ in a range of scenarios that occur randomly in applications where exact knowledge about $\{|A_1|, \ldots, |A_M|\}$ with respect to $\sigma^2$ is not or only roughly available. Naively fixing the test length to some value $N$ based on a certain assumed operating point can result in a strongly variable performance in practical scenarios. Additionally, if $N$ is conservatively set to a high value based on the worst still acceptable operating point, a lot of time spent for detection of the best beam will be wasted, if the channel quality is actually better than expected. This is particularly important when the channel coherence time is limited and waisting time for training results in loss of throughput. This gives us the motivation to look for an adaptive variable length test that can achieve the same desirable performance over a range of operating points while keeping the test length (i.e. training time) as small as possible.

III. GENERALIZED LIKELIHOOD RATIO TEST: DETECTION OF SIGNAL PRESENCE

The building block of our sequential test presented in the next section is the fixed length Generalized Likelihood Ratio Test (GLRT) to detect the presence or absence of the signal with unknown amplitude.

Consider the following composite detection problem

$$\mathcal{H}_0 : y[n] = w[n]$$
$$\mathcal{H}_1 : y[n] = A + w[n] ,$$

where $n = 1, \ldots, N$, $A$ is a nonzero complex deterministic unknown amplitude and $w[n]$ is zero mean complex WGN with unknown variance $\sigma^2$.

Since the variance of the PDF under $\mathcal{H}_0$ is not known, a proper threshold to bound the probability of deciding $\mathcal{H}_1$ when $\mathcal{H}_0$ is true, typically denoted as the probability of false alarm $P_{\text{FA}}$, can not be found using a simple Neyman-Pearson (NP) approach [7]. Instead one can use the Maximum Likelihood (ML) estimates of the unknown parameters derived from the available observations and insert them into the likelihood functions under each hypothesis. The ML estimates of $A$ and $\sigma^2$ under $\mathcal{H}_1$ are $\hat{A} = \bar{y} T / N = \bar{y}$ (the sample mean) and $\hat{\sigma}^2_{\mathcal{H}_1} = (|y - \bar{y}|^2) / N$, while under $\mathcal{H}_0$ the ML estimate of $\sigma^2$ is just $\hat{\sigma}^2_{\mathcal{H}_0} = (|y|^2) / N$.

By replacing the unknown parameters with their ML estimates in the Gaussian PDFs, the GLR can be calculated as

$$L_{\text{G}}(y) = \frac{p(y; \hat{A}, \hat{\sigma}^2_{\mathcal{H}_1}, \mathcal{H}_1)}{p(y; \hat{\sigma}^2_{\mathcal{H}_0}, \mathcal{H}_0)} = \left(\frac{\hat{\sigma}^2_{\mathcal{H}_0}}{\hat{\sigma}^2_{\mathcal{H}_1}}\right)^{\frac{N}{2}} .$$

Fig. 2. Illustration of $\chi^2$ distributions with different $\lambda$ values.

Fig. 3. Decay of $P_{\text{MD}}$ with increasing test length $N$ for different SNR values where $P_{\text{FA}} = 0.1$ for solid lines and $P_{\text{FA}} = 0.01$ for dashed lines.

Eq. (5) indicates that deciding for $\mathcal{H}_1$ makes sense when the fit of the data to the signal amplitude estimate $\hat{A} = \bar{y}$ produces a smaller error, as measured by $\hat{\sigma}_{\mathcal{H}_1}^2$, compared to the fit to the no signal hypothesis reflected by the estimate $\hat{\sigma}_{\mathcal{H}_0}^2$.

The remaining task is to find a proper decision threshold in order to bound $P_{\text{FA}}$. Let us define the modified GLR statistic

$$\gamma = 2 \ln L_{\text{G}}(y) .$$

A non-trivial result that can be found in [8] states that for large $N$ the random variable $\gamma$ follows either a central or a non-central $\chi^2$-distribution with one degree of freedom so that

$$\gamma = N \ln \left(\frac{\hat{\sigma}^2_{\mathcal{H}_0}}{\hat{\sigma}^2_{\mathcal{H}_1}}\right) \sim \left\{ \begin{array}{ll}
\chi^2_1, & \text{under } \mathcal{H}_0 \ \text{or} \ \mathcal{H}_1,
\end{array} \right.$$

The ratio $\lambda = N(|A|^2/\sigma^2)$ that plays the role of the deflection coefficient is denoted in statistics as the non-centrality parameter of the $\chi^2_1(\lambda)$-PDF (see Fig. 2). Since now the PDF of $\gamma$ under $\mathcal{H}_0$ is completely known, using the NP design criterion, we can ensure that $P_{\text{FA}}$ will not surpass a predefined value by finding a proper threshold $\gamma_{\text{th}}$. Noting that a $\chi^2_1$ r.v. $\gamma$ is related to a standard normal r.v. $x \sim N(0, 1)$ as $\gamma = x^2$, it follows that $P_{\text{FA}} = \Pr(\gamma > \gamma_{\text{th}}; \mathcal{H}_0)$ can be expressed as a sum of $Q$-functions so that $P_{\text{FA}} = \Pr(x > \sqrt{\gamma_{\text{th}}}) + \Pr(x < -\sqrt{\gamma_{\text{th}}}) = 2Q(\sqrt{\gamma_{\text{th}}})$. This specifies $\gamma_{\text{th}}$ in terms of $P_{\text{FA}}$ as

$$\gamma_{\text{th}} = \left[Q^{-1}(P_{\text{FA}})\right]^2 .$$

Similarly, the second type of error, the probability of misdetection $P_{\text{MD}} = 1 - \Pr(\gamma > \gamma_{\text{th}}; \mathcal{H}_1)$ can be stated in closed form as

$$P_{\text{MD}} = Q(\sqrt{\lambda} - \sqrt{\gamma_{\text{th}}}) - Q(\sqrt{\lambda} + \sqrt{\gamma_{\text{th}}}).$$

It is worth noting that, for large enough $N$, the test

$$\gamma_{\text{th}} \approx \gamma_{\text{th}}$$

(9)
is a Uniformly Most Powerful (UMP) test. This means the performance of this detector in terms of $P_{MD}$ achieves the bound given by the clairvoyant NP detector that assumes perfect knowledge of the non-centrality parameter $\gamma$ under $H_1$.

As shown in Fig. 3, the probability of misdirection $P_{MD}$ decreases exponentially as $N$ increases due to the $Q$-function. In addition, the rate of decrease depends strongly on SNR per observation $|A|^2/\sigma^2$. The sequential competition test that we introduce next exploits this strong dependence of $P_{MD}$ on $|A|^2/\sigma^2$ and $N$ (while $P_{FA}$ is fixed), and turns it into an advantage.

IV. SEQUENTIAL COMPETITION TEST

Consider again the initial $M$-ary decision problem stated in Section II where separate observation sequences corresponding to different beams are available and the aim is to detect the beam that captures the highest signal magnitude. This time, instead of comparing the estimates of the values $\{A_1, \ldots, A_M\}$ directly against each other as in Eq. (2), let us rather compare them separately to the absence of a signal. This means that under each beam $m \in \{1, \ldots, M\}$ we formulate the same binary hypothesis test stated in the previous section as

$$
\mathcal{H}_{m,0}: \ y_m[n] = w_m[n]
$$

$$
\mathcal{H}_{m,1}: \ y_m[n] = A_m + w_m[n]
$$

(10)

However, this time $n$ can grow until a decision criterion is fulfilled. Obviously, $\mathcal{H}_{m,0}$ is the wrong hypothesis under each beam, assuming that some signal is observable but with different strength. On the other hand, $\mathcal{H}_{m,1}$ acts as a virtual common reference in the set of $M$ parallel binary tests, into which the $M$-ary test is decomposed.

Let us denote the probability of selecting the presence of the signal in the binary test of beam $m$ after $n$ observations as $P_{\mathcal{H}_{m,1}}(n)$. Comparing the binary tests of beam $m$ and beam $m'$, it follows from Eq. (8) that for $|A_m| > |A_{m'}|$ and a common decision threshold $\gamma_m$ that $P_{\mathcal{H}_{m,1}}(n) > P_{\mathcal{H}_{m',1}}(n)$. This is simply a consequence of the fact that the accumulated deflection coefficient $\lambda_m = n|A_m|^2/\sigma^2$ will grow more quickly than $\lambda_{m'} = n|A_{m'}|^2/\sigma^2$. Therefore, if we use the modified GLR statistic introduced in the last section as a decision metric, the beam that observes the stronger signal will on average cross the threshold earlier. This fact leads to the following sequential competition test applied to stochastic paths $\gamma_m[n]$ for $m = 1, \ldots, M$ stated with pseudo code in Algorithm 1.

At each step $n$ all stochastic paths $\gamma_m[n]$ corresponding to beams $m = 1, \ldots, M$ are compared to the fixed common threshold $\gamma_0$. The test terminates as soon as one of the paths surpasses the threshold while the index of this path indicates the selected beam. Otherwise, we continue by taking the next observation into account. The interpretation is that we let the beams compete to distinguish themselves from pure zero mean WGN with the same unknown variance, and the one which does it faster is the winning beam in the competition. The test length $n$ is now a random variable with average $\bar{n}$.

**Algorithm 1 Sequential Competition Test**

1. **input:** $M, P_{FA}, N_{max}, n = 1, \gamma_m[1] = 0$
2. $\gamma_0 = [Q^{-1}(P_{FA}/2)]^2$
3. while $\max_m(\gamma_m[n]) < \gamma_0$ and $n \leq N_{max}$ do
   4. $n = n + 1$
5. for $m = 1, \ldots, M$ do
   6. $y_m = \{y_m[1], \ldots, y_m[n]\}$
   7. $q_m^n = \sum_{t=0}^{n} y_m[t]/n$
   8. $\delta_{\gamma_m}^2 = \|y_mq_m^n\|/n$
   9. $\delta_{\gamma_m}^2 = ([y_m - \bar{y}_m][y_m - \bar{y}_m])/\|q_m^n\|$
   10. $\gamma_m[n] = n \ln(\delta_{\gamma_m}^2/\delta_{\gamma_m}^2)$
6. end for
9. end while
13. **output:** $\arg\max_m(\gamma_m[n])$

The common threshold $\gamma_0$ for all virtual binary tests is selected using Eq. (7) without knowing the values $\{\gamma_m, \sigma^2\}$. The higher we put the threshold (i.e. choosing a smaller allowed $P_{FA}$ under each virtual binary test) the longer it takes on average for the sequential competition test to terminate and make a decision. This however, leads to a more accurate discrimination of the strongest beam among all competitors and therefore better performance in terms of $\bar{l}$.

In the following toy example depicted in Fig. 4 with $M = 2$ and ratio $r = |A_1|/|A_2| = 0.5$, we observe that the sequential competition test shows an essentially invariant adaptive performance in terms of $\bar{l}$ for varying value of the differential SNR between the candidate amplitudes. This is in strong contrast to the fixed length test that uses length of $N_{fix} = 25$. The reason is that the average test length $\bar{n}$ (over multiple realizations) of the sequential competition test adapts itself to $\bar{l}$ for varying value of the differential SNR between the candidate amplitudes. This is in strong contrast to the fixed length test that uses length of $N_{fix} = 25$. The reason is that the average test length $\bar{n}$ (over multiple realizations) of the sequential competition test adapts itself to $\bar{l}$ for varying value of the differential SNR between the candidate amplitudes. This is in strong contrast to the fixed length test that uses length of $N_{fix} = 25$.
more time is left for data transmission. On the other hand, the more time we spend for training, the loss of SNR due to inaccurate training will reduce. This means for $N_{\text{max}}$ possible channel uses within finite transmission time, we can evaluate the ratio between effective data rates by sequential and fixed length tests via

$$\frac{R_{\text{eff,seq}}}{R_{\text{eff,fix}}} = \frac{\mathbb{E}\left[(1 - \frac{n}{N_{\text{max}}}) \log(1 + (1 - l_{\text{seq}})\frac{|A_{\text{max}}|^2}{\sigma^2})\right]}{\mathbb{E}\left[(1 - \frac{n}{N_{\text{max}}}) \log(1 + (1 - l_{\text{fix}})\frac{|A_{\text{max}}|^2}{\sigma^2})\right]}, \quad (11)$$

where $l_{\text{seq}}$ and $l_{\text{fix}}$ are normalized signal magnitude loss under each individual simulation run and $\mathbb{E}$ denotes the expectation over all simulation runs. As depicted in Fig. 6 bottom, the sequential competition test fulfills the training-transmission trade-off better compared to fixed length tests by providing higher effective data rate at almost all SNR operating points.

V. NUMERICAL EVALUATION

We numerically studied the performance of the sequential competition test in the reference channel model described in Eq. (1) with a uniform linear array with 16 antenna elements using the codebook of a Butler matrix with 16 beams. The AoA was distributed uniformly in $[-90^\circ, 90^\circ]$ over the simulation runs while SNR was defined as $|A_{\text{max}}|^2/\sigma^2$ indicating the maximum available SNR of the best beam. The quantities $\bar{l}$ and $\bar{n}$ were estimated at each SNR point based on $10^4$ simulation runs for SNR values in the interval $[-8, 8]$ dB. For comparison we consider the fixed length detector based on MVU power estimates stated in Eq. 2, with $N_{\text{fix}} \in [10, 25, 50, 75]$. As shown in Fig. 6, the sequential competition test with the same $\gamma_n$ for all beams based on $P_{\text{FA}} = 10^{-3}$ keeps $\bar{l}$ in an interval of $[0.07, 0.14]$ considered to be sufficient in practice, while adaptively decreasing the average test length $\bar{n}$ as $|A_{\text{max}}|^2/\sigma^2$ grows larger. In contrast, fixed length tests show strongly variable performance. For instance, the fixed length test with $N_{\text{fix}} = 25$ results in $\bar{l}$ in an interval of $[0.005, 0.055]$.

In case of finite transmission time (e.g. limited channel coherence time), the shorter we spend time for training, the

![Fig. 4. Comparing the achieved $\bar{l}$ between a fixed length test with length $N_{\text{fix}}$ and the sequential test with equal $\gamma_n$ based on $P_{\text{FA}} = 10^{-3}$.](image)

![Fig. 5. Performance of the sequential test over ratio $r$. The upper-bound (beam selection by coin tossing) on $\bar{l}$ is depicted as black dot dashed line.](image)

![Fig. 6. Achieved $\bar{l}$, $\bar{n}$ and ratio between effective rates using sequential competition test and fixed length tests with $N_{\text{max}} = 150$.](image)
REFERENCES


